CHAPTER 1
Limits and Their Properties

Section 1.1 A Preview of Calculus
Solutions to Odd-Numbered Exercises

1. Precalculus: (20 ft/sec)(15 seconds) = 300 feet

3. Calculus required: slope of tangent line at \( x = 2 \) is rate of change, and equals about 0.16.

5. Precalculus: Area = \( \frac{1}{2}bh = \frac{1}{2}(5)(3) = \frac{15}{2} \) sq. units

7. Precalculus: Volume = \( (2)(4)(3) = 24 \) cubic units

9. (a)

![Graph of a function]

(b) The graphs of \( y_2 \) are approximations to the tangent line to \( y_1 \) at \( x = 1 \).
(c) The slope is approximately 2. For a better approximation make the list numbers smaller:
\{0.2, 0.1, 0.01, 0.001\}

11. (a) \( D_1 = \sqrt{(5 - 1)^2 + (1 - 5)^2} = \sqrt{16 + 16} \approx 5.66 \)
(b) \( D_2 = \sqrt{1 + \left(\frac{3}{2}\right)^2} + \sqrt{1 + \left(\frac{5}{4} - \frac{3}{2}\right)^2} + \sqrt{1 + \left(\frac{5}{4} - \frac{3}{2}\right)^2} + \sqrt{1 + \left(\frac{1}{4} - 1\right)^2} \approx 2.693 + 1.302 + 1.083 + 1.031 \approx 6.11 \)
(c) Increase the number of line segments.

Section 1.2 Finding Limits Graphically and Numerically

1.

<table>
<thead>
<tr>
<th>( x )</th>
<th>1.9</th>
<th>1.99</th>
<th>1.999</th>
<th>2.001</th>
<th>2.01</th>
<th>2.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>0.3448</td>
<td>0.3344</td>
<td>0.3334</td>
<td>0.3332</td>
<td>0.3322</td>
<td>0.3226</td>
</tr>
</tbody>
</table>

\[ \lim_{x \to 2} \frac{x - 2}{x^2 - x - 2} \approx 0.3333 \quad (\text{Actual limit is} \quad \frac{1}{4}) \]

3.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-0.1</th>
<th>-0.01</th>
<th>-0.001</th>
<th>0.001</th>
<th>0.01</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>0.2911</td>
<td>0.2889</td>
<td>0.2887</td>
<td>0.2887</td>
<td>0.2884</td>
<td>0.2863</td>
</tr>
</tbody>
</table>

\[ \lim_{x \to 0} \frac{\sqrt{x + 3} - \sqrt{3}}{x} = 0.2887 \quad (\text{Actual limit is} \quad \frac{1}{2\sqrt{3}}) \]
5. Actual limit is \( \frac{1}{x^2} \). (Actual limit is \( \frac{1}{x^2} \).)

\[
\lim_{x \to 3} \frac{1}{x^2} = -0.0625 \quad \text{lim}_{x \to 3} \frac{1}{x^2} = -0.0625 \quad \text{(Actual limit is } \frac{1}{x^2} \text{.)}
\]

\[
\begin{array}{ccccccc}
 x & 2.9 & 2.99 & 2.999 & 3.001 & 3.01 & 3.1 \\
 f(x) & -0.0641 & -0.0627 & -0.0625 & -0.0625 & -0.0623 & -0.0610 \\
\end{array}
\]

\[
\lim_{x \to 3} \frac{1}{x^2} = -0.0625 \quad \text{lim}_{x \to 3} \frac{1}{x^2} = -0.0625 \quad \text{(Actual limit is } \frac{1}{x^2} \text{.)}
\]

7. \( \lim_{x \to 0} \sin \frac{x}{x} = 1.0000 \) (Actual limit is 1.) (Make sure you use radian mode.)

\[
\begin{array}{ccccccc}
 x & -0.1 & -0.01 & -0.001 & 0.001 & 0.01 & 0.1 \\
 f(x) & 0.9983 & 0.99998 & 1.0000 & 1.0000 & 0.99998 & 0.9983 \\
\end{array}
\]

\[
\lim_{x \to 0} \sin \frac{x}{x} = 1.0000 \quad \text{lim}_{x \to 0} \sin \frac{x}{x} = 1.0000 \quad \text{(Actual limit is } 1.0000 \text{.) (Make sure you use radian mode.)}
\]

9. \( \lim_{x \to 3} (4 - x) = 1 \)

11. \( \lim_{x \to 2} f(x) = \lim_{x \to 2} (4 - x) = 2 \)

13. \( \lim_{x \to 5} \frac{|x - 5|}{x - 5} \) does not exist. For values of \( x \) to the left of 5, \( \frac{|x - 5|}{x - 5} \) equals \(-1\), whereas for values of \( x \) to the right of 5, \( \frac{|x - 5|}{x - 5} \) equals \(1\).

15. \( \lim_{x \to \pi/2} \tan x \) does not exist since the function increases and decreases without bound as \( x \) approaches \( \pi/2 \).

17. \( \lim_{x \to 0} \cos(1/x) \) does not exist since the function oscillates between \(-1\) and \(1\) as \( x \) approaches \(0\).

19. \( C(t) = 0.75 - 0.50 \|t - 1\| \)

(a) \[
\begin{array}{c}
\begin{array}{c}
0 \\
\vdots \\
3 \\
\vdots \\
5 \\
\end{array}
\end{array}
\]

(b) \[
\begin{array}{cccccccc}
 t & 3 & 3.3 & 3.4 & 3.5 & 3.6 & 3.7 & 4 \\
 C & 1.75 & 2.25 & 2.25 & 2.25 & 2.25 & 2.25 & 2.25 \\
\end{array}
\]

\[
\lim_{t \to 3.5} C(t) = 2.25 
\]

(c) \[
\begin{array}{cccccccc}
 t & 2 & 2.5 & 2.9 & 3 & 3.1 & 3.5 & 4 \\
 C & 1.25 & 1.75 & 1.75 & 2.25 & 2.25 & 2.25 & 2.25 \\
\end{array}
\]

\[
\lim_{t \to 3} C(t) \text{ does not exist. The values of } C \text{ jump from } 1.75 \text{ to } 2.25 \text{ at } t = 3. 
\]

21. You need to find \( \delta \) such that \( 0 < |x - 1| < \delta \) implies \( |f(x) - 1| = \left| \frac{1}{x} - 1 \right| < 0.1 \). That is,

\[
-0.1 < \frac{1}{x} - 1 < 0.1
\]

\[
1 - 0.1 < \frac{1}{x} < 1 + 0.1
\]

\[
\frac{9}{10} < \frac{1}{x} < \frac{11}{10}
\]

\[
\frac{10}{9} > x > \frac{10}{11}
\]

\[
\frac{10}{9} - 1 > x - 1 > \frac{10}{11} - 1
\]

\[
\frac{1}{9} > x - 1 > \frac{1}{11}
\]

So take \( \delta = \frac{1}{11} \). Then \( 0 < |x - 1| < \delta \) implies

\[
-\frac{1}{11} < x - 1 < \frac{1}{11}
\]

\[
\frac{1}{11} < x - 1 < \frac{1}{9}
\]

Using the first series of equivalent inequalities, you obtain

\[
|f(x) - 1| = \left| \frac{1}{x} - 1 \right| < \varepsilon < 0.1.
\]

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23. \( \lim_{x \to 2} (3x + 2) = 8 = L \)
   \(|(3x + 2) - 8| < 0.01\)
   \(3|x - 2| < 0.01\)
   \(0 < |x - 2| < \frac{0.01}{3} \approx 0.0033 = \delta \)
   Hence, if \(0 < |x - 2| < \delta = \frac{0.01}{3}\), you have
   \(3|x - 2| < 0.01\)
   \(|3x - 6| < 0.01\)
   \(|(3x + 2) - 8| < 0.01\)
   \(|f(x) - L| < 0.01\)

25. \( \lim_{x \to 2} (x^2 - 3) = 1 = L \)
   \(|(x^2 - 3) - 1| < 0.01\)
   \(|x^2 - 4| < 0.01\)
   \(|(x + 2)(x - 2)| < 0.01\)
   \(|x + 2| |x - 2| < 0.01\)
   \(0 < |x - 2| < \frac{0.01}{|x + 2|}\)
   If we assume \(1 < x < 3\), then \(\delta = 0.01/5 = 0.002\).
   Hence, if \(0 < |x - 2| < \delta = 0.002\), you have
   \(|x - 2| < 0.002 = \frac{4}{5}(0.01) < \frac{1}{|x + 2|}(0.01)\)
   \(|x + 2| |x - 2| < 0.01\)
   \(|x^2 - 4| < 0.01\)
   \(|(x^2 - 3) - 1| < 0.01\)
   \(|f(x) - L| < 0.01\)

27. \( \lim_{x \to 2} (x + 3) = 5 \)
   Given \(\epsilon > 0\):
   \(|(x + 3) - 5| < \epsilon\)
   \(|x - 2| < \epsilon = \delta\)
   Hence, let \(\delta = \epsilon\).
   Hence, if \(0 < |x - 2| < \delta = \epsilon\), you have
   \(|x - 2| < \epsilon\)
   \(|(x + 3) - 5| < \epsilon\)
   \(|f(x) - L| < \epsilon\)

29. \( \lim_{x \to -4} \left( \frac{1}{2} x - 1 \right) = \frac{1}{2}(-4) - 1 = -3 \)
   Given \(\epsilon > 0\):
   \(\left| \left( \frac{1}{2} x - 1 \right) - (-3) \right| < \epsilon\)
   \(\left| \frac{1}{2} x + 2 \right| < \epsilon\)
   \(\frac{1}{2} |x - (-4)| < \epsilon\)
   \(|x - (-4)| < 2\epsilon\)
   Hence, let \(\delta = 2\epsilon\).
   Hence, if \(0 < |x - (-4)| < \delta = 2\epsilon\), you have
   \(|x - (-4)| < 2\epsilon\)
   \(\left| \frac{1}{2} x + 2 \right| < \epsilon\)
   \(\left| \left( \frac{1}{2} x - 1 \right) + 3 \right| < \epsilon\)
   \(|f(x) - L| < \epsilon\)

31. \( \lim_{x \to 6} 3 = 3 \)
   Given \(\epsilon > 0\):
   \(|3 - 3| < \epsilon\)
   \(0 < \epsilon\)
   Hence, any \(\delta > 0\) will work.
   Hence, for any \(\delta > 0\), you have
   \(|3 - 3| < \epsilon\)
   \(|f(x) - L| < \epsilon\)

33. \( \lim_{x \to 0} \sqrt{x} = 0 \)
   Given \(\epsilon > 0\):
   \(\sqrt{|x - 0|} < \epsilon\)
   \(|\sqrt{x}| < \epsilon\)
   \(|x| < \epsilon^2 = \delta\)
   Hence, let \(\delta = \epsilon^2\).
   Hence for \(0 < |x - 0| < \delta = \epsilon^2\), you have
   \(|x| < \epsilon^2\)
   \(|\sqrt{x} - 0| < \epsilon\)
   \(|f(x) - L| < \epsilon\)
35. \( \lim_{x \to -2} |x - 2| = |(-2) - 2| = 4 \)

Given \( \varepsilon > 0 \):

\[ |x - 2| - 4 < \varepsilon \]
\[ -(x - 2) - 4 < \varepsilon \quad (x - 2 < 0) \]
\[ -x - 2 = |x + 2| = |x - (-2)| < \varepsilon \]

Hence, \( \delta = \varepsilon \).

Hence for \( 0 < |x - 2| < \delta = \varepsilon \), you have

\[ |x + 2| < \varepsilon \]
\[ |-(x + 2)| < \varepsilon \]
\[ -(x - 2) - 4 < \varepsilon \]
\[ ||x - 2| - 4| < \varepsilon \quad \text{(because } x - 20) \]
\[ |f(x) - L| < \varepsilon \]

39. \( f(x) = \frac{\sqrt{x + 5} - 3}{x - 4} \)

\( \lim_{x \to 4} f(x) = \frac{1}{6} \)

41. \( f(x) = \frac{x - 9}{\sqrt{x} - 3} \)

\( \lim_{x \to 9} f(x) = 6 \)

The domain is \([ -5, 4] \cup (4, \infty) \).

The graphing utility does not show the hole at \((4, \frac{1}{6})\).

The domain is all \( x \geq 0 \) except \( x = 9 \). The graphing utility does not show the hole at \((9, 6)\).

37. \( \lim_{x \to 1} (x^2 + 1) = 2 \)

Given \( \varepsilon > 0 \):

\[ |(x^2 + 1) - 2| < \varepsilon \]
\[ |x^2 - 1| < \varepsilon \]
\[ |(x + 1)(x - 1)| < \varepsilon \]
\[ |x - 1| < \frac{\varepsilon}{2} < \frac{1}{|x + 1|} \]
\[ |x^2 - 1| < \varepsilon \]
\[ |(x^2 + 1) - 2| < \varepsilon \]
\[ |f(x) - 2| < \varepsilon \]

43. \( \lim_{x \to 8} f(x) = 25 \) means that the values of \( f \) approach 25 as \( x \) gets closer and closer to 8.

45. (i) The values of \( f \) approach different numbers as \( x \) approaches \( c \) from different sides of \( c \):

(ii) The values of \( f \) increase without bound as \( x \) approaches \( c \):

(iii) The values of \( f \) oscillate between two fixed numbers as \( x \) approaches \( c \):

47. \( f(x) = (1 + x)^{1/3} \)

\( \lim_{x \to 0} (1 + x)^{1/3} = e = 2.71828 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
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<tbody>
<tr>
<td>-0.1</td>
<td>2.867972</td>
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<td>-0.01</td>
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<td>2.719642</td>
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<tr>
<td>-0.0001</td>
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<td>-0.00001</td>
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<tr>
<td>-0.000001</td>
<td>2.718283</td>
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<table>
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<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>2.593742</td>
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<tr>
<td>0.01</td>
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<tr>
<td>0.000001</td>
<td>2.718280</td>
</tr>
</tbody>
</table>
49. False; \( f(x) = \frac{\sin x}{x} \) is undefined when \( x = 0 \).
   From Exercise 7, we have
   \[
   \lim_{x \to 0} \frac{\sin x}{x} = 1.
   \]

51. False; let
   \[
   f(x) = \begin{cases} 
   x^2 - 4x, & x \neq 4 \\
   10, & x = 4
   \end{cases}
   \]
   \[
   f(4) = 10
   \]
   \[
   \lim_{x \to 4} f(x) = \lim_{x \to 4} (x^2 - 4x) = 0 \neq 10
   \]

53. Answers will vary.

55. If \( \lim_{x \to c} f(x) = L_1 \) and \( \lim_{x \to c} f(x) = L_2 \), then for every \( \epsilon > 0 \), there exists \( \delta_1 > 0 \) and \( \delta_2 > 0 \) such that \( |x - c| < \delta_1 \implies |f(x) - L_1| < \epsilon \) and \( |x - c| < \delta_2 \implies |f(x) - L_2| < \epsilon \). Let \( \delta \) equal the smaller of \( \delta_1 \) and \( \delta_2 \). Then for \( |x - c| < \delta \), we have
   \[
   |L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| \leq |L_1 - f(x)| + |f(x) - L_2| < \epsilon + \epsilon.
   \]
   Therefore, \( |L_1 - L_2| < 2\epsilon \). Since \( \epsilon > 0 \) is arbitrary, it follows that \( L_1 = L_2 \).

57. \( \lim_{x \to c} [f(x) - L] = 0 \) means that for every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that if
   \[
   0 < |x - c| < \delta,
   \]
   then
   \[
   |(f(x) - L) - 0| < \epsilon.
   \]
   This means the same as \( |f(x) - L| < \epsilon \) when
   \[
   0 < |x - c| < \delta.
   \]
   Thus, \( \lim_{x \to c} f(x) = L \).

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### Section 1.3 Evaluating Limits Analytically

1. \( \lim_{x \to 5} h(x) = 0 \)  
   \[
   h(x) = x^2 - 5x
   \]
2. \( \lim_{x \to 1} h(x) = 6 \)  
3. \( \lim_{x \to 0} f(x) = 0 \)  
   \[
   f(x) = \cos x
   \]
4. \( \lim_{x \to \pi/3} f(x) \approx 0.524 \)  
   \[
   = \frac{\pi}{6}
   \]
5. \( \lim_{x \to 2} x^4 = 16 \)
6. \( \lim_{x \to 0} (2x - 1) = 2(0) - 1 = -1 \)
7. \( \lim_{x \to 3} (x^2 + 3x) = (-3)^2 + 3(-3) = 9 - 9 = 0 \)
8. \( \lim_{x \to 3} (2x^2 + 4x + 1) = 2(-3)^2 + 4(-3) + 1 = 18 - 12 + 1 = 7 \)
9. \( \lim_{x \to 2} \frac{1}{x} = \frac{1}{2} \)
10. \( \lim_{x \to 4} x^2 + 4 = \frac{1}{2^2} + 4 = 1 \frac{3}{2} = \frac{5}{2} \)
11. \( \lim_{x \to 5} \frac{5x}{\sqrt{x} + 2} = \frac{5(7)}{\sqrt{7} + 2} = \frac{35}{\sqrt{9}} = \frac{35}{3} \)
12. \( \lim_{x \to 3} \sqrt{x + 1} = \sqrt{3 + 1} = 2 \)
21. \( \lim_{x \to -4} (x + 3)^2 = (-4 + 3)^2 = 1 \)

23. (a) \( \lim_{x \to 1} f(x) = 5 - 1 = 4 \)
   
   (b) \( \lim_{x \to 4} g(x) = 4^3 = 64 \)
   
   (c) \( \lim_{x \to 1} g(f(x)) = g(f(1)) = g(4) = 64 \)

25. (a) \( \lim_{x \to 4} f(x) = 4 - 1 = 3 \)
   
   (b) \( \lim_{x \to 3} g(x) = \sqrt{3 + 1} = 2 \)
   
   (c) \( \lim_{x \to 1} g(f(x)) = g(3) = 2 \)

27. \( \lim_{x \to 0} \sin x = \sin \frac{\pi}{2} = 1 \)

29. \( \lim_{x \to 2} \cos \frac{\pi x}{3} = \cos \frac{2\pi}{3} = -\frac{1}{2} \)

31. \( \lim_{x \to 0} \sec 2x = \sec 0 = 1 \)

33. \( \lim_{x \to 5\pi/6} \sin x = \sin \frac{5\pi}{6} = \frac{1}{2} \)

37. (a) \( \lim_{x \to 3} [5g(x)] = 5 \lim_{x \to 3} g(x) = 5(3) = 15 \)
   
   (b) \( \lim_{x \to 3} [f(x) + g(x)] = \lim_{x \to 3} f(x) + \lim_{x \to 3} g(x) = 2 + 3 = 5 \)
   
   (c) \( \lim_{x \to 4} [f(x)g(x)] = [\lim_{x \to 4} f(x)][\lim_{x \to 4} g(x)] = (2)(3) = 6 \)
   
   (d) \( \lim_{x \to 3} \frac{f(x)}{g(x)} = \lim_{x \to 3} \frac{f(x)}{\lim_{x \to 3} g(x)} = \frac{2}{3} \)

41. \( f(x) = -2x + 1 \) and \( g(x) = -\frac{2x^2 + x}{x} \) agree except at \( x = 0 \).
   
   (a) \( \lim_{x \to 0} g(x) = \lim_{x \to 0} f(x) = 1 \)
   
   (b) \( \lim_{x \to 1} g(x) = \lim_{x \to 1} f(x) = 3 \)

45. \( f(x) = \frac{x^2 - 1}{x + 1} \) and \( g(x) = x - 1 \) agree except at \( x = -1 \).
   
   \( \lim_{x \to -1} f(x) = \lim_{x \to -1} g(x) = -2 \)

49. \( \lim_{x \to 5} \frac{x - 5}{x^2 - 25} = \lim_{x \to 5} \frac{x - 5}{(x + 5)(x - 5)} \)
   
   \( = \lim_{x \to 5} \frac{1}{x + 5} = \frac{1}{10} \)

51. \( \lim_{x \to 3} \frac{x^2 + x - 6}{x^3 - 9} = \lim_{x \to 3} \frac{(x + 3)(x - 2)}{(x + 3)(x^2 - 3x)} \)
   
   \( = \lim_{x \to 3} \frac{x - 2}{x - 3} = \frac{-5}{-6} = \frac{5}{6} \)
53. \[ \lim_{x \to 0} \frac{\sqrt{x+5} - \sqrt{5}}{x} = \lim_{x \to 0} \frac{\sqrt{x+5} - \sqrt{5}}{x} \cdot \frac{\sqrt{x+5} + \sqrt{5}}{\sqrt{x+5} + \sqrt{5}} \]
\[ = \lim_{x \to 0} \frac{(x+5) - 5}{x(\sqrt{x+5} + \sqrt{5})} = \lim_{x \to 0} \frac{1}{\sqrt{x+5} + \sqrt{5}} = \frac{1}{2\sqrt{5}} = \frac{\sqrt{5}}{10} \]

55. \[ \lim_{x \to 4} \frac{\sqrt{x+5} - 3}{x - 4} = \lim_{x \to 4} \frac{\sqrt{x+5} - 3}{x - 4} \cdot \frac{\sqrt{x+5} + 3}{\sqrt{x+5} + 3} \]
\[ = \lim_{x \to 4} \frac{(x+5) - 9}{(x-4)(\sqrt{x+5} + 3)} = \lim_{x \to 4} \frac{1}{\sqrt{x+5} + 3} = \frac{1}{\sqrt{9} + 3} = \frac{1}{6} \]

57. \[ \lim_{x \to 0} \frac{2 + x - \frac{1}{2}}{x} = \lim_{x \to 0} \frac{2 - (2 + x)}{x} = \lim_{x \to 0} \frac{-1}{2(2 + x)} = -\frac{1}{4} \]

59. \[ \lim_{\Delta x \to 0} \frac{2(x + \Delta x) - 2x}{\Delta x} = \lim_{\Delta x \to 0} \frac{2x + 2\Delta x - 2x}{\Delta x} = \lim_{\Delta x \to 0} 2 = 2 \]

61. \[ \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - 2(x + \Delta x) + 1 - (x^2 - 2x + 1)}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - 2x - 2\Delta x + 1 - x^2 + 2x - 1}{\Delta x} \]
\[ = \lim_{\Delta x \to 0} (2x + \Delta x - 2) = 2x - 2 \]

63. \[ \lim_{x \to 0} \frac{\sqrt{x+2} - \sqrt{2}}{x} = 0.354 \]

<table>
<thead>
<tr>
<th>(x)</th>
<th>-0.1</th>
<th>-0.01</th>
<th>-0.001</th>
<th>0</th>
<th>0.001</th>
<th>0.01</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(x))</td>
<td>0.358</td>
<td>0.354</td>
<td>0.345</td>
<td>?</td>
<td>0.354</td>
<td>0.353</td>
<td>0.349</td>
</tr>
</tbody>
</table>

Analytically, \[ \lim_{x \to 0} \frac{\sqrt{x+2} - \sqrt{2}}{x} = \lim_{x \to 0} \frac{\sqrt{x+2} - \sqrt{2}}{x} \cdot \frac{\sqrt{x+2} + \sqrt{2}}{\sqrt{x+2} + \sqrt{2}} \]
\[ = \lim_{x \to 0} \frac{x+2-2}{x(\sqrt{x+2} + \sqrt{2})} = \lim_{x \to 0} \frac{1}{\sqrt{x+2} + \sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4} \approx 0.354 \]

65. \[ \lim_{x \to 0} \frac{2 + x - \frac{1}{2}}{x} = \frac{1}{4} \]

<table>
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<tr>
<th>(x)</th>
<th>-0.1</th>
<th>-0.01</th>
<th>-0.001</th>
<th>0</th>
<th>0.001</th>
<th>0.01</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(x))</td>
<td>-0.263</td>
<td>-0.251</td>
<td>-0.250</td>
<td>?</td>
<td>-0.250</td>
<td>-0.249</td>
<td>-0.238</td>
</tr>
</tbody>
</table>

Analytically, \[ \lim_{x \to 0} \frac{2 + x - \frac{1}{2}}{x} = \lim_{x \to 0} \frac{2 - (2 + x)}{2(2 + x)} \cdot \frac{1}{x} = \lim_{x \to 0} \frac{-x}{2(2 + x)} \cdot \frac{1}{x} = \lim_{x \to 0} \frac{-1}{2(2 + x)} = \frac{-1}{4} \]
67. \( \lim_{x \to 0} \frac{\sin x}{5x} = \lim_{x \to 0} \left[ \frac{(\sin x)(1/5)}{x} \right] = (1)\left(\frac{1}{5}\right) = \frac{1}{5} \)

69. \( \lim_{x \to 0} \frac{\sin x(1 - \cos x)}{2x^2} = \lim_{x \to 0} \left[ \frac{1}{2} \cdot \frac{\sin x - \sin x}{x} \cdot \frac{1 - \cos x}{x} \right] = \frac{1}{2} \lim_{x \to 0} (1)(0) = 0 \)

71. \( \lim_{x \to 0} \frac{\sin^2 x}{x} = \lim_{x \to 0} \left[ \frac{\sin x}{x} \cdot \sin x \right] = (1)\sin 0 = 0 \)

73. \( \lim_{h \to 0} \frac{(1 - \cos h)^2}{h} = \lim_{h \to 0} \left[ \frac{1 - \cos h}{h} \cdot (1 - \cos h) \right] = (0)(0) = 0 \)

75. \( \lim_{x \to \pi/2} \frac{\cos x}{\cot x} = \lim_{x \to \pi/2} \sin x = 1 \)

77. \( \lim_{t \to 0} \frac{\sin 3t}{2t} = \lim_{t \to 0} \left( \frac{\sin 3t}{3t} \right) \left( \frac{3}{2} \right) = (1)\left(\frac{3}{2}\right) = \frac{3}{2} \)

The limit appear to equal 3.

79. \( f(t) = \frac{\sin 3t}{t} \)

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\text{t} & -0.1 & -0.01 & -0.001 & 0 & 0.001 & 0.1 \\
\hline
f(t) & 2.96 & 2.9996 & 3 & ? & 3 & 2.9996 & 2.96 \\
\hline
\end{array}
\]

Analytically, \( \lim_{t \to 0} \frac{\sin 3t}{t} = \lim_{t \to 0} \left( \frac{\sin 3t}{3t} \right) = 3(1) = 3 \).

81. \( f(x) = \frac{\sin x^2}{x} \)

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\text{x} & -0.1 & -0.01 & -0.001 & 0 & 0.001 & 0.1 \\
\hline
f(x) & -0.099998 & -0.01 & -0.001 & ? & 0.001 & 0.099998 \\
\hline
\end{array}
\]

Analytically, \( \lim_{x \to 0} \frac{\sin x^2}{x} = \lim_{x \to 0} \left( \frac{\sin x^2}{x^2} \right) = 0(1) = 0 \).

83. \( \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{2(x + h) + 3 - (2x + 3)}{h} = \lim_{h \to 0} \frac{2x + 2h + 3 - 2x - 3}{h} = \lim_{h \to 0} \frac{2h}{h} = 2 \)

85. \( \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{4 + \frac{4}{x + h} - \frac{4}{x}}{h} = \lim_{h \to 0} \frac{4x - 4(x + h)}{(x + h)x} = \lim_{h \to 0} \frac{-4}{x^2} = -\frac{4}{x^2} \)

87. \( \lim_{x \to 0} (4 - x^2) \leq \lim_{x \to 0} f(x) \leq \lim_{x \to 0} (4 + x^2) \)

\( 4 \leq \lim_{x \to 0} f(x) \leq 4 \)

Therefore, \( \lim_{x \to 0} f(x) = 4 \).

89. \( f(x) = x \cos x \)

\[
\lim_{x \to 0} (x \cos x) = 0
\]
91. \( f(x) = |x| \sin x \)

\[
\lim_{x \to 0} |x| \sin x = 0
\]

95. We say that two functions \( f \) and \( g \) agree at all but one point (on an open interval) if \( f(x) = g(x) \) for all \( x \) in the interval except for \( x = c \), where \( c \) is in the interval.

97. An indeterminant form is obtained when evaluating a limit using direct substitution produces a meaningless fractional expression such as \( 0/0 \). That is,

\[
\lim_{x \to c} \frac{f(x)}{g(x)}
\]

for which \( \lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0 \)

99. \( f(x) = x, \ g(x) = \sin x, \ h(x) = \frac{\sin x}{x} \)

When you are “close to” 0 the magnitude of \( f \) is approximately equal to the magnitude of \( g \). Thus, \( |\sin x|/|x| = 1 \) when \( x \) is “close to” 0.

101. \( s(t) = -16t^2 + 1000 \)

\[
\lim_{t \to 3} \frac{s(5) - s(t)}{5 - t} = \lim_{t \to 3} \frac{600 - (-16t^2 + 1000)}{5 - t} = \lim_{t \to 3} \frac{16(t + 5)(t - 5)}{5 - t} = \lim_{t \to 3} -16(t + 5) = -160 \text{ ft/sec.}
\]

Speed = 160 ft/sec

103. \( s(t) = -4.9t^2 + 150 \)

\[
\lim_{t \to 3} \frac{s(3) - s(t)}{3 - t} = \lim_{t \to 3} \frac{-4.9(3^2) + 150 - (-4.9t^2 + 150)}{3 - t} = \lim_{t \to 3} \frac{-4.9(9 - t^2)}{3 - t} = \lim_{t \to 3} \frac{-4.9(3 - t)(3 + t)}{3 - t} = \lim_{t \to 3} -4.9(3 + t) = -29.4 \text{ m/sec}
\]

105. Let \( f(x) = 1/x \) and \( g(x) = -1/x \). \( \lim_{x \to 0^+} f(x) \) and \( \lim_{x \to 0^-} g(x) \) do not exist.

\[
\lim_{x \to 0^+} [f(x) + g(x)] = \lim_{x \to 0^+} \left[ \frac{1}{x} + \left( -\frac{1}{x} \right) \right] = \lim_{x \to 0^+} [0] = 0
\]

107. Given \( f(x) = b \), show that for every \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that \( |f(x) - b| < \epsilon \) whenever \( |x - c| < \delta \). Since \( |f(x) - b| = |b - b| = 0 < \epsilon \) for any \( \epsilon > 0 \), then any value of \( \delta > 0 \) will work.

109. If \( b = 0 \), then the property is true because both sides are equal to 0. If \( b \neq 0 \), let \( \epsilon > 0 \) be given. Since \( \lim_{x \to c} f(x) = L \), there exists \( \delta > 0 \) such that \( |f(x) - L| < \epsilon/|b| \) whenever \( 0 < |x - c| < \delta \). Hence, wherever \( 0 < |x - c| < \delta \), we have

\[
|b||f(x) - L| < \epsilon \text{ or } |bf(x) - bL| < \epsilon
\]

which implies that \( \lim_{x \to c} [bf(x)] = bL \).
111. \[ -M[f(x)] \leq f(x)g(x) \leq M[f(x)] \]
\[ \lim_{x \to c} (-M[f(x)]) \leq \lim_{x \to c} f(x)g(x) \leq \lim_{x \to c} (M[f(x)]) \]
\[ -M(0) \leq \lim_{x \to c} f(x)g(x) \leq M(0) \]
\[ 0 \leq \lim_{x \to c} f(x)g(x) \leq 0 \]
Therefore, \[ \lim_{x \to c} f(x)g(x) = 0. \]

113. False. As \( x \) approaches 0 from the left, \[ \frac{|x|}{x} = -1. \]

115. True.

117. False. The limit does not exist.

119. Let
\[ f(x) = \begin{cases} 
4, & \text{if } x \geq 0 \\
-4, & \text{if } x < 0 
\end{cases} \]
\[ \lim_{x \to 0} |f(x)| = \lim_{x \to 0} 4 = 4. \]
\[ \lim_{x \to 0} f(x) \text{ does not exist since for } x < 0, f(x) = -4 \text{ and for } x \geq 0, f(x) = 4. \]

121. \( f(x) = \begin{cases} 
0, & \text{if } x \text{ is rational} \\
1, & \text{if } x \text{ is irrational} 
\end{cases} \)
\( g(x) = \begin{cases} 
0, & \text{if } x \text{ is rational} \\
x, & \text{if } x \text{ is irrational} 
\end{cases} \)
\[ \lim_{x \to 0} f(x) \text{ does not exist.} \]
No matter how “close to” 0 \( x \) is, there are still an infinite number of rational and irrational numbers so that \( \lim_{x \to 0} f(x) \) does not exist.
\[ \lim_{x \to 0} g(x) = 0. \]
When \( x \) is “close to” 0, both parts of the function are “close to” 0.

123. (a) \[ \lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{1 - \cos x}{x^2} \cdot \frac{1 + \cos x}{1 + \cos x} \]
\[ = \lim_{x \to 0} \frac{1 - \cos^2 x}{x^2(1 + \cos x)} \]
\[ = \lim_{x \to 0} \frac{\sin^2 x}{x^2} \cdot \frac{1}{1 + \cos x} \]
\[ = \left(1 \cdot \frac{1}{2}\right) = \frac{1}{2} \]
(b) Thus, \[ \frac{1 - \cos x}{x^2} \approx \frac{1}{2} \Rightarrow 1 - \cos x \approx \frac{1}{2} x^2 \]
\[ \Rightarrow \cos x \approx 1 - \frac{1}{2} x^2 \text{ for } x = 0. \]
(c) \( \cos(0.1) \approx 1 - \frac{1}{2}(0.1)^2 = 0.995 \)
(d) \( \cos(0.1) \approx 0.9950, \) which agrees with part (c).
Section 1.4  Continuity and One-Sided Limits

1. (a) \( \lim_{x \to 3} f(x) = 1 \)  \hspace{2cm} 3. (a) \( \lim_{x \to 0} f(x) = 0 \)  \hspace{2cm} 5. (a) \( \lim_{x \to 1} f(x) = 2 \)
(b) \( \lim_{x \to 3} f(x) = 1 \)  \hspace{1cm} (b) \( \lim_{x \to 3} f(x) = 0 \)  \hspace{1cm} (b) \( \lim_{x \to 4} f(x) = -2 \)
(c) \( \lim_{x \to 3} f(x) = 1 \)  \hspace{1cm} (c) \( \lim_{x \to 3} f(x) = 0 \)  \hspace{1cm} (c) \( \lim_{x \to 4} f(x) \) does not exist

The function is continuous at \( x = 3 \).

The function is NOT continuous at \( x = 3 \).

The function is NOT continuous at \( x = 4 \).

7. \( \lim_{x \to 3} \frac{x - 5}{x^2 - 25} = \lim_{x \to 3} \frac{1}{x + 5} = \frac{1}{10} \)

9. \( \lim_{x \to -3} \frac{x}{\sqrt{x^2 - 9}} \) does not exist because \( \frac{x}{\sqrt{x^2 - 9}} \) grows without bound as \( x \to -3^- \).

11. \( \lim_{x \to 0} \frac{|x|}{x} = \lim_{x \to 0} \frac{-x}{x} = -1. \)

13. \( \lim_{\Delta x \to 0} \frac{x + \Delta x}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \cdot \frac{x}{x + \Delta x} = \lim_{\Delta x \to 0} \frac{-\Delta x}{x(x + \Delta x)} = \frac{1}{x^2} \)

15. \( \lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x + 2}{2} = \frac{5}{2} \)

17. \( \lim_{x \to -1} \frac{f(x)}{x^3 + 1} = \lim_{x \to -1} \frac{(x + 1)}{x^3 + 1} = \lim_{x \to -1} (x + 1) = 2 \)

19. \( \lim_{x \to \pi} \cot x \) does not exist since

\( \lim_{x \to \pi^-} \cot x \) and \( \lim_{x \to \pi^+} \cot x \) do not exist.

21. \( \lim_{x \to 4} (3[x] - 5) = 3(3) - 5 = 4 \)

23. \( \lim_{x \to 3} (2 - \lfloor x \rfloor) \) does not exist because

\( \lim_{x \to 3} (2 - \lfloor x \rfloor) = 2 - (-3) = 5 \) and

\( \lim_{x \to 3} (2 - \lfloor x \rfloor) = 2 - (-4) = 6. \)

25. \( f(x) = \frac{1}{x^2 - 4} \) has discontinuities at \( x = -2 \) and \( x = 2 \) since \( f(-2) \) and \( f(2) \) are not defined.

27. \( f(x) = \frac{\|x\|}{2} + x \) has discontinuities at each integer \( k \) since \( \lim_{x \to k} f(x) \neq \lim_{x \to k^+} f(x). \)

29. \( g(x) = \sqrt{25 - x^2} \) is continuous on \([-5, 5]\).

31. \( \lim_{x \to 0^+} f(x) = 3 = \lim_{x \to 1^-} f(x). \) \( f \) is continuous on \([-1, 4]\).

33. \( f(x) = x^2 - 2x + 1 \) is continuous for all real \( x \).
35. \( f(x) = 3x - \cos x \) is continuous for all real \( x \).

37. \( f(x) = \frac{x}{x^2 - x} \) is not continuous at \( x = 0, 1 \). Since
\[
\frac{x}{x^2 - x} = \frac{1}{x - 1}
\]
for \( x \neq 0, x = 0 \) is a removable discontinuity, whereas \( x = 1 \) is a nonremovable discontinuity.

39. \( f(x) = \frac{x}{x^2 + 1} \) is continuous for all real \( x \).

41. \( f(x) = \frac{x + 2}{(x + 2)(x - 5)} \) has a nonremovable discontinuity at \( x = 5 \) since \( \lim_{x \to 5} f(x) \) does not exist, and has a removable discontinuity at \( x = -2 \) since
\[
\lim_{x \to -2} f(x) = \lim_{x \to -2} \frac{1}{x - 5} = -\frac{1}{7}.
\]

43. \( f(x) = \frac{|x + 2|}{x + 2} \) has a nonremovable discontinuity at \( x = -2 \) since \( \lim_{x \to -2} f(x) \) does not exist.

45. \( f(x) = \begin{cases} x, & x \leq 1 \\ x^2, & x > 1 \end{cases} \) has a possible discontinuity at \( x = 1 \).

1. \( f(1) = 1 \)

2. \( \lim_{x \to 1} f(x) = \lim_{x \to 1} x = 1 \)

Thus \( \lim_{x \to 1} f(x) = 1 \)

3. \( f(1) = \lim_{x \to 1} f(x) \)

\( f \) is continuous at \( x = 1 \), therefore, \( f \) is continuous for all real \( x \).

47. \( f(x) = \begin{cases} \frac{x}{2} + 1, & x \leq 2 \\ \frac{3 - x}{2}, & x > 2 \end{cases} \) has a possible discontinuity at \( x = 2 \).

1. \( f(2) = \frac{2}{2} + 1 = 2 \)

\[ \lim_{x \to 2} f(x) = \lim_{x \to 2} \left( \frac{x}{2} + 1 \right) = 2 \]

\( \lim_{x \to 2} f(x) \) does not exist.

2. \( \lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{3 - x}{2} = 1 \)

Therefore, \( f \) has a nonremovable discontinuity at \( x = 2 \).

49. \( f(x) = \begin{cases} \frac{\tan \frac{\pi x}{4}}{x}, & |x| < 1 \\ \frac{\tan \frac{\pi x}{4}}{x}, & |x| \geq 1 \end{cases} \) has possible discontinuities at \( x = -1, x = 1 \).

1. \( f(-1) = -1 \) \quad \( f(1) = 1 \)

2. \( \lim_{x \to -1} f(x) = -1 \) \quad \( \lim_{x \to 1} f(x) = 1 \)

3. \( f(-1) = \lim_{x \to -1} f(x) \) \quad \( f(1) = \lim_{x \to 1} f(x) \)

\( f \) is continuous at \( x = \pm 1 \), therefore, \( f \) is continuous for all real \( x \).
51. \( f(x) = \csc 2x \) has nonremovable discontinuities at integer multiples of \( \pi/2 \).

55. \( \lim_{x \to 0^+} f(x) = 0 \)

\[ \lim_{x \to 0^-} f(x) = 0 \]

\( f \) is not continuous at \( x = -2 \).

57. \( f(2) = 8 \)

Find \( a \) so that \( \lim_{x \to 2^-} ax^2 = 8 \Rightarrow a = \frac{8}{2^2} = 2 \).

59. Find \( a \) and \( b \) such that \( \lim_{x \to -1^+} (ax + b) = -a + b = 2 \) and \( \lim_{x \to 3^-} (ax + b) = 3a + b = -2 \).

\[ a - b = -2 \]

\[ (+) \quad 3a + b = -2 \]

\[ 4a = -4 \]

\[ a = -1 \]

\[ b = 2 + (-1) = 1 \]

61. \( f(g(x)) = (x - 1)^2 \)

Continuous for all real \( x \).

65. \( y = [x] - x \)

Nonremovable discontinuity at each integer

69. \( f(x) = \frac{x}{x^2 + 1} \)

Continuous on \((-\infty, \infty)\)

71. \( f(x) = \sec \frac{\pi x}{4} \)

Continuous on \((-\infty, \infty)\): \(-6, \ldots, -2, 2, 6, 10, \ldots\)

73. \( f(x) = \frac{\sin x}{x} \)

The graph appears to be continuous on the interval \([-4, 4]\). Since \( f(0) \) is not defined, we know that \( f \) has a discontinuity at \( x = 0 \). This discontinuity is removable so it does not show up on the graph.

53. \( f(x) = [x - 1] \) has nonremovable discontinuities at each integer \( k \).

57. \( f(2) = 8 \)

Find \( a \) so that \( \lim_{x \to 2^-} ax^2 = 8 \Rightarrow a = \frac{8}{2^2} = 2 \).
77. \( f(x) = x^3 - 2 - \cos x \) is continuous on \([0, \pi]\).

\[ f(0) = -3 \text{ and } f(\pi) = -1 > 0. \]

By the Intermediate Value Theorem, \( f(c) = 0 \) for the least one value of \( c \) between 0 and \( \pi \).

79. \( f(x) = x^3 + x - 1 \)

\( f(x) \) is continuous on \([0, 1]\).

\[ f(0) = -1 \text{ and } f(1) = 1 \]

By the Intermediate Value Theorem, \( f(x) = 0 \) for at least one value of \( c \) between 0 and 1. Using a graphing utility, we find that \( x \approx 0.6823 \).

81. \( g(t) = 2 \cos t - 3t \)

\( g \) is continuous on \([0, 1]\).

\[ g(0) = 2 > 0 \text{ and } g(1) = -1.9 < 0. \]

By the Intermediate Value Theorem, \( g(t) = 0 \) for at least one value \( c \) between 0 and 1. Using a graphing utility, we find that \( t \approx 0.5636 \).

83. \( f(x) = x^2 + x - 1 \)

\( f \) is continuous on \([0, 5]\).

\[ f(0) = -1 \text{ and } f(5) = 29 \]

\[ -1 < 11 < 29 \]

The Intermediate Value Theorem applies.

\[ x^2 + x - 1 = 11 \]
\[ x^2 + x - 12 = 0 \]
\[ (x + 4)(x - 3) = 0 \]
\[ x = -4 \text{ or } x = 3 \]

\( c = 3 \) (\( x = -4 \) is not in the interval.)

Thus, \( f(3) = 11 \).

85. \( f(x) = x^3 - x^2 + x - 2 \)

\( f \) is continuous on \([0, 3]\).

\[ f(0) = -2 \text{ and } f(3) = 19 \]

\[-2 < 4 < 19 \]

The Intermediate Value Theorem applies.

\[ x^3 - x^2 + x - 2 = 4 \]
\[ x^3 - x^2 + x - 6 = 0 \]
\[ (x - 2)(x^2 + x + 3) = 0 \]
\[ x = 2 \]
\[ (x^2 + x + 3 \text{ has no real solution.}) \]
\[ c = 2 \]

Thus, \( f(2) = 4 \).

87. (a) The limit does not exist at \( x = c \).

(b) The function is not defined at \( x = c \).

(c) The limit exists at \( x = c \), but it is not equal to the value of the function at \( x = c \).

(d) The limit does not exist at \( x = c \).

89. 

The function is not continuous at \( x = 3 \) because \( \lim_{x \to 3^+} f(x) = 1 \neq 0 = \lim_{x \to 3^-} f(x) \).

91. The functions agree for integer values of \( x \):

\[ g(x) = 3 - \lfloor -x \rfloor = 3 - (x) = 3 + x \] \text{ for } \( x \) an integer

\[ f(x) = 3 + \lfloor x \rfloor = 3 + x \]

However, for non-integer values of \( x \), the functions differ by 1.

\[ f(x) = 3 + \lfloor x \rfloor = g(x) - 1 = 2 - \lfloor -x \rfloor. \]

For example, \( f(\frac{1}{2}) = 3 + 0 = 3, g(\frac{1}{2}) = 3 - (-1) = 4 \).
93. \[ N(t) = 25 \left( 2 \left\lfloor \frac{t + 2}{2} \right\rfloor - t \right) \]

<table>
<thead>
<tr>
<th>( t )</th>
<th>0</th>
<th>1</th>
<th>1.8</th>
<th>2</th>
<th>3</th>
<th>3.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N(t) )</td>
<td>50</td>
<td>25</td>
<td>5</td>
<td>50</td>
<td>25</td>
<td>5</td>
</tr>
</tbody>
</table>

Discontinuous at every positive even integer. The company replenishes its inventory every two months.

95. Let \( V = \frac{4}{3} \pi r^3 \) be the volume of a sphere of radius \( r \)

\[ V(1) = \frac{4}{3} \pi \approx 4.19 \]
\[ V(5) = \frac{4}{3} \pi (5^3) = 523.6 \]

Since \( 4.19 < 275 < 523.6 \), the Intermediate Value Theorem implies that there is at least one value \( r \) between 1 and 5 such that \( V(r) = 275 \). (In fact, \( r \approx 4.0341 \).)

97. Let \( c \) be any real number. Then \( \lim_{x \to c} f(x) \) does not exist since there are both rational and irrational numbers arbitrarily close to \( c \). Therefore, \( f \) is not continuous at \( c \).

99. \( \text{sgn}(x) = \begin{cases} -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \end{cases} \)

(a) \( \lim_{x \to 0^-} \text{sgn}(x) = -1 \)
(b) \( \lim_{x \to 0^+} \text{sgn}(x) = 1 \)
(c) \( \lim_{x \to 0} \text{sgn}(x) \) does not exist.

101. True; if \( f(x) = g(x), x \neq c \), then \( \lim_{x \to c} f(x) = \lim_{x \to c} g(x) \) and at least one of these limits (if they exist) does not equal the corresponding function at \( x = c \).

103. False; \( f(1) \) is not defined and \( \lim_{x \to 1} f(x) \) does not exist.

105. (a) \( f(x) = \begin{cases} 0 & \text{if } 0 \leq x < b \\ b & \text{if } b \leq x \leq 2b \end{cases} \)

(b) \( g(x) = \begin{cases} \frac{x}{2} & \text{if } 0 \leq x \leq b \\ b - \frac{x}{2} & \text{if } b < x \leq 2b \end{cases} \)

NOT continuous at \( x = b \).
107. \( f(x) = \frac{\sqrt{x + c^2} - c}{x} \), \( c > 0 \)

Domain: \( x + c^2 \geq 0 \implies x \geq -c^2 \) and \( x \neq 0 \), \([-c^2, 0) \cup (0, \infty)\)

\[
\lim_{x \to 0} \frac{\sqrt{x + c^2} - c}{x} = \lim_{x \to 0} \frac{\sqrt{x + c^2} - c}{x} \cdot \frac{\sqrt{x + c^2} + c}{\sqrt{x + c^2} + c} = \lim_{x \to 0} \frac{1}{\sqrt{x + c^2} + c} = \frac{1}{2c}
\]

Define \( f(0) = 1/(2c) \) to make \( f \) continuous at \( x = 0 \).

109. \( h(x) = x \| x \| \)

\( h \) has nonremovable discontinuities at

\[ x = \pm 1, \pm 2, \pm 3, \ldots \]

---

**Section 1.5 Infinite Limits**

1. \( \lim_{x \to -2} 2 \left| \frac{x}{x^2 - 4} \right| = \infty \)

2. \( \lim_{x \to -2} 2 \left| \frac{x}{x^2 - 4} \right| = \infty \)

3. \( \lim_{x \to \frac{\pi}{4}} \tan \frac{\pi x}{4} = -\infty \)

4. \( \lim_{x \to \frac{\pi}{4}} \tan \frac{\pi x}{4} = \infty \)

5. \( f(x) = \frac{1}{x^2 - 9} \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>3.5</th>
<th>3.1</th>
<th>3.01</th>
<th>3.001</th>
<th>-2.999</th>
<th>-2.99</th>
<th>-2.9</th>
<th>-2.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>0.308</td>
<td>1.639</td>
<td>16.64</td>
<td>166.6</td>
<td>-166.7</td>
<td>-16.69</td>
<td>-1.695</td>
<td>-0.364</td>
</tr>
</tbody>
</table>

\( \lim_{x \to -3} f(x) = \infty \)

\( \lim_{x \to -3} f(x) = -\infty \)

6. \( f(x) = \frac{x^2}{x^2 - 9} \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>3.5</th>
<th>3.1</th>
<th>3.01</th>
<th>3.001</th>
<th>-2.999</th>
<th>-2.99</th>
<th>-2.9</th>
<th>-2.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>3.769</td>
<td>15.75</td>
<td>150.8</td>
<td>1501</td>
<td>-1499</td>
<td>-149.3</td>
<td>-142.5</td>
<td>-2.273</td>
</tr>
</tbody>
</table>

\( \lim_{x \to -3} f(x) = \infty \)

\( \lim_{x \to -3} f(x) = -\infty \)
9. \( \lim_{x \to 10} \frac{1}{x} = \infty = \lim_{x \to 10} \frac{1}{x} \)

Therefore, \( x = 0 \) is a vertical asymptote.

11. \( \lim_{x \to 2} \frac{x^2 - 2}{(x - 2)(x + 1)} = \infty \)

\( \lim_{x \to 2} \frac{x^2 - 2}{(x - 2)(x + 1)} = -\infty \)

Therefore, \( x = 2 \) is a vertical asymptote.

13. \( \lim_{x \to -2} \frac{x^2}{x^2 - 4} = \infty \) and \( \lim_{x \to -2} \frac{x^2}{x^2 - 4} = -\infty \)

Therefore, \( x = -2 \) is a vertical asymptote.

15. No vertical asymptote since the denominator is never zero.

17. \( f(x) = \tan 2x = \frac{\sin 2x}{\cos 2x} \) has vertical asymptotes at

\( x = \frac{(2n + 1) \pi}{4} = \frac{\pi}{4} + \frac{n \pi}{2}, n \) any integer.

19. \( \lim_{t \to 0^+} \left( 1 - \frac{4}{t^2} \right) = -\infty = \lim_{t \to 0^+} \left( 1 - \frac{4}{t^2} \right) \)

Therefore, \( t = 0 \) is a vertical asymptote.

21. \( \lim_{x \to 2^+} \frac{x}{(x + 2)(x - 1)} = \infty \)

\( \lim_{x \to 2^-} \frac{x}{(x + 2)(x - 1)} = -\infty \)

Therefore, \( x = -2 \) is a vertical asymptote.

23. \( f(x) = \frac{x^3 + 1}{x + 1} = \frac{(x + 1)(x^2 - x + 1)}{x + 1} \)

has no vertical asymptote since

\( \lim_{x \to -1} f(x) = \lim_{x \to -1} (x^2 - x + 1) = 3 \)

25. \( f(x) = \frac{(x - 5)(x + 3)}{(x - 5)(x^2 + 1)} = \frac{x + 3}{x^2 + 1}, x \neq 5 \)

No vertical asymptotes. The graph has a hole at \( x = 5 \).

27. \( s(t) = \frac{t}{\sin t} \) has vertical asymptotes at \( t = n \pi, n \) a nonzero integer. There is no vertical asymptote at \( t = 0 \) since

\( \lim_{t \to 0} \frac{t}{\sin t} = 1. \)
29. \( \lim_{x \to 1} \frac{x^2 - 1}{x + 1} = \lim_{x \to 1} (x - 1) = -2 \)

Removable discontinuity at \( x = -1 \)

33. \( \lim_{x \to -2} \frac{x - 3}{x - 2} = -\infty \)

37. \( \lim_{x \to -3} \frac{x^2 + 2x - 3}{x^3 + x - 6} = \lim_{x \to -3} \frac{x - 1}{x - 2} = \frac{4}{5} \)

41. \( \lim_{x \to 0^+} \left( 1 + \frac{1}{x} \right) = -\infty \)

45. \( \lim_{x \to \pi} \frac{\sqrt{x}}{\csc x} = \lim_{x \to \pi} \left( \sqrt{x} \sin x \right) = 0 \)

49. \( f(x) = \frac{x^2 + x + 1}{x^3 - 1} \)

\( \lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{1}{x - 1} = \infty \)

53. A limit in which \( f(x) \) increases or decreases without bound as \( x \) approaches \( c \) is called an infinite limit. \( \infty \) is not a number. Rather, the symbol

\( \lim_{x \to c} f(x) = \infty \)

says how the limit fails to exist.

55. One answer is \( f(x) = \frac{x - 3}{(x - 6)(x + 2)} = \frac{x - 3}{x^2 - 4x - 12} \)

59. \( S = \frac{k}{1 - r}, 0 < |r| < 1 \). Assume \( k \neq 0 \).

\( \lim_{r \to 1^-} S = \lim_{r \to 1^-} \frac{k}{1 - r} = \infty \) (or \(-\infty \) if \( k < 0 \))
61. \( C = \frac{528x}{100 - x}, 0 \leq x < 100 \)

(a) \( C(25) = 176 \text{ million} \)

(b) \( C(50) = 528 \text{ million} \)

(c) \( C(75) = 1584 \text{ million} \)

(d) \( \lim_{x \to 100} \frac{528}{100 - x} = \infty \) Thus, it is not possible.

63. (a) \( r = \frac{2(7)}{\sqrt{625 - 49}} = \frac{7}{12} \text{ ft/sec} \)

(b) \( r = \frac{2(15)}{\sqrt{625 - 225}} = \frac{3}{2} \text{ ft/sec} \)

(c) \( \lim_{x \to 25} \frac{2x}{\sqrt{625 - x^2}} = \infty \)

65. (a) 

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>0.5</th>
<th>0.2</th>
<th>0.1</th>
<th>0.01</th>
<th>0.001</th>
<th>0.0001</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>0.1585</td>
<td>0.0411</td>
<td>0.0067</td>
<td>0.0017</td>
<td>( \approx 0 )</td>
<td>( \approx 0 )</td>
<td>( \approx 0 )</td>
</tr>
</tbody>
</table>

\[ \lim_{x \to 0} \frac{x - \sin x}{x} = 0 \]

(b) 

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
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<th>0.2</th>
<th>0.1</th>
<th>0.01</th>
<th>0.001</th>
<th>0.0001</th>
</tr>
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<tbody>
<tr>
<td>( f(x) )</td>
<td>0.1585</td>
<td>0.0823</td>
<td>0.0333</td>
<td>0.0167</td>
<td>0.0017</td>
<td>( \approx 0 )</td>
<td>( \approx 0 )</td>
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</tbody>
</table>

\[ \lim_{x \to 0} \frac{x - \sin x}{x^2} = 0 \]

(c) 

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>0.5</th>
<th>0.2</th>
<th>0.1</th>
<th>0.01</th>
<th>0.001</th>
<th>0.0001</th>
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</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>0.1585</td>
<td>0.1646</td>
<td>0.1663</td>
<td>0.1666</td>
<td>0.1667</td>
<td>0.1667</td>
<td>0.1667</td>
</tr>
</tbody>
</table>

\[ \lim_{x \to 0} \frac{x - \sin x}{x^3} = 0.1167 \ (1/6) \]

(d) 

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>0.5</th>
<th>0.2</th>
<th>0.1</th>
<th>0.01</th>
<th>0.001</th>
<th>0.0001</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>0.1585</td>
<td>0.3292</td>
<td>0.8317</td>
<td>1.6658</td>
<td>16.67</td>
<td>166.7</td>
<td>1667.0</td>
</tr>
</tbody>
</table>

\[ \lim_{x \to 0} \frac{x - \sin x}{x^4} = \infty \]

For \( n \geq 3 \), \( \lim_{x \to 0} \frac{x - \sin x}{x^n} = \infty \).
67. (a) Because the circumference of the motor is half that of the saw arbor, the saw makes 1700/2 = 850 revolutions per minute.

(c) \(2(20 \cot \phi) + 2(10 \cot \phi)\): straight sections.

The angle subtended in each circle is

\[
2\pi - \left(2\left(\frac{\pi}{2} - \phi\right)\right) = \pi + 2\phi.
\]

Thus, the length of the belt around the pulleys is

\[
20(\pi + 2\phi) + 10(\pi + 2\phi) = 30(\pi + 2\phi).
\]

Total length = 60 cot \(\phi + 30(\pi + 2\phi)\)

Domain: \(\left(0, \frac{\pi}{2}\right)\)

69. False; for instance, let

\[
f(x) = \frac{x^2 - 1}{x - 1}
\]
or

\[
g(x) = \frac{x}{x^2 + 1}.
\]

73. Given \(\lim_{x \to a} f(x) = \infty\) and \(\lim_{x \to a} g(x) = L:\)

(2) Product:

If \(L > 0\), then for \(\varepsilon = L/2 > 0\) there exists \(\delta_1 > 0\) such that \(|g(x) - L| < L/2\) whenever \(0 < |x - c| < \delta_1\). Thus, \(L/2 < g(x) < 3L/2\). Since \(\lim_{x \to a} f(x) = \infty\) then for \(M > 0\), there exists \(\delta_2 > 0\) such that \(f(x) > M(2/L)\) whenever \(0 < |x - c| < \delta_2\). Let \(\delta = \min\{\delta_1, \delta_2\}\). Then for \(0 < |x - c| < \delta\), we have \(f(x)g(x) > M(2/L)(L/2) = M\).

Therefore \(\lim_{x \to a} f(x)g(x) = \infty\). The proof is similar for \(L < 0\).

(3) Quotient: Let \(\varepsilon > 0\) be given.

There exists \(\delta_1 > 0\) such that \(f(x) > 3L/2\varepsilon\) whenever \(0 < |x - c| < \delta_1\) and there exists \(\delta_2 > 0\) such that \(|g(x) - L| < L/2\) whenever \(0 < |x - c| < \delta_2\). This inequality gives us \(L/2 < g(x) < 3L/2\). Let \(\delta = \min\{\delta_1, \delta_2\}\). Then for \(0 < |x - c| < \delta\), we have

\[
\left|\frac{g(x)}{f(x)}\right| < \frac{3L/2}{3L/2\varepsilon} = \varepsilon.
\]

Therefore, \(\lim_{x \to a} g(x)/f(x) = 0\).

75. Given \(\lim_{x \to a} f(x) = 0\).

Suppose \(\lim f(x)\) exists and equals \(L\). Then,

\[
\lim_{x \to a} \frac{1}{f(x)} = \lim_{x \to a} \frac{1}{\lim_{x \to a} f(x)} = \frac{1}{L} = 0.
\]

This is not possible. Thus, \(\lim f(x)\) does not exist.